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ON THE DISSIPATIVE RESPONSE DUE TO DISCONTINUOUS STRAINS IN BARS OF UNSTABLE ELASTIC MATERIAL

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Abstract—Some elastic materials are capable of sustaining finite equilibrium deformations with discontinuous strains. Boundary-value problems for such "unstable" elastic materials often possess an infinite number of solutions, suggesting that the theory suffers from a constitutive deficiency. In the setting of the one-dimensional theory of bars in tension, the present paper explores the consequences of supplementing the theory with further constitutive information. This additional information pertains to the surface of strain discontinuity and consists of a "kinetic relation" and a criterion for the "initiation" of such a surface. We show that the quasi-static response of the bar to a prescribed force history is then fully determined. In particular, we observe how unstable elastic materials can be used to model macroscopic behavior similar to that associated with viscoplasticity.

1. INTRODUCTION

Bodies composed of certain types of homogeneous elastic materials can be finitely deformed to equilibrium states in which displacement gradients, strains and stresses suffer jump discontinuities across special surfaces. Elastostatic fields of this kind arise, for example, in continuum mechanical treatments of stress-induced phase transformations in solids (James, 1979, 1984).

When such jumps in displacement gradient occur during quasi-static, isothermal motions, the balance between the rate of increase of stored energy and the rate of work of external forces associated with conventionally smooth deformations of elastic bodies no longer holds. This balance is replaced by one which includes an additional effect that may be interpreted as the rate of work of a fictitious "driving traction" acting on the moving surface of discontinuity (Knowles, 1979). The driving traction is formally related to the notion of a "force on a defect" introduced by Eshelby (1956) and discussed by Rice (1975).

The altered energetics of finite elastostatic fields involving strain jumps suggest that such fields might be used to model certain types of dissipative behavior in solids. The circumstances, in fact, are reminiscent in some respects of those present in the classical theory of flows of ideal fluids in which shock waves are present. In the latter subject, shocks account in an idealized way for the neglected dissipative effects of viscosity and heat conduction (see p. 322 of Landau and Lifshitz (1959)). Because of this similarity, we refer to surfaces bearing jump discontinuities in the displacement gradient in an elastostatic field as "equilibrium shocks".

Not all elastic materials are capable of sustaining deformations with equilibrium shocks. Those that do have this capability are sometimes called unstable materials; they lead to differential equations of equilibrium that necessarily fail to be elliptic at *some* deformation (Knowles and Sternberg, 1978). This in turn leads to a massive loss of uniqueness of solution for the boundary value problems of elastostatics, suggesting the need for additional constitutive assumptions that will select from among the many possible equilibrium states that one which is preferred by the body. One such additional constitutive postulate asserts that the material is conservative at all of its particles, including those on shocks, so that the body prefers the equilibrium state which renders the appropriate energy functional an absolute minimum. With this assumption in force, the driving traction acting on any equilibrium shock necessarily vanishes (Abeyaratne, 1983; Gurtin, 1983), the conventional balance between work and energy is preserved, and no dissipation takes place. In this conservative setting, elastostatic fields with shocks have recently received much analytical attention (James, 1979, 1981, 1986; Gurtin, 1983; Ericksen, 1975; Abeyaratne, 1980; Ball and James, 1987; Fosdick and James, 1981; Fosdick and MacSithigh, 1983; Silling, 1988).

In two recent papers (Abeyaratne and Knowles, 1987a, b), we have discussed an example in order to illustrate an alternative constitutive postulate. The problem treated in these papers involves a finite, plane deformation of an infinite medium containing a circular cavity. A uniform circumferential traction is applied to the cavity wall, and the displacement is required to vanish at infinity. For the class of incompressible, isotropic elastic materials considered, the resulting twisting deformation may exhibit a circular equilibrium shock concentric with the cavity. In quasi-static motions of the body involving such equilibrium states, the relationship between the applied torque and the twist at the cavity wall—i.e. the macroscopic response—is in general hysteretic. We showed that if a certain maximum-dissipation postulate is used as the supplementary constitutive assumption, the macroscopic response mimics that associated with rate-independent elastic-plastic behavior.

Our purpose in the present paper is to discuss supplementary constitutive models for elastic fields capable of sustaining equilibrium shocks in more generality. The principal new feature introduced here is a "kinetic relation" analogous to those arising in microstructural models of plastic behavior formulated in terms of internal variables (Rice, 1970, 1971, 1975). In our circumstances, this relation takes the form of a constitutive law connecting the driving traction acting on a moving shock with the shock velocity during a quasi-static, isothermal motion. We show that appropriate choices of the kinetic relation lead to visco-plastic macroscopic response, and we recover conservative (minimum-energy) response as well as rate-independent elastic–plastic behavior as special or limiting cases.

When there is an equilibrium shock in the body, the kinetic law governs its evolution. However a separate criterion—an initiation or nucleation criterion—is required in order to signal the initial appearance of the shock. This too will be discussed in the following.

For simplicity, we work here in the context of a one-dimensional model for extensional deformations of an elastic bar. Our setting is thus essentially that of Ericksen (1975) in his discussion of one-dimensional deformations with strain jumps, except that we consider bars the cross-sectional area of which varies with position. The special case of the *uniform* bar turns out to be exceptional in certain important respects.

After introducing in the following section the class of elastic materials to be considered, we investigate equilibrium states with a single shock in Section 3. Sections 4 and 5 are concerned with the energetics of quasi-static, isothermal motions of the bar and the admissibility of such motions according to the second law of thermodynamics. We introduce the notion of a kinetic relation as well as a shock initiation criterion in Section 6, and in Section 7 present examples to illustrate the possibilities offered by the theory.

An approach of the type put forward here may have application to the modeling of the mechanical response of shape-memory alloys (Delaey *et al.*, 1974), to continuum descriptions of the effect of the presence of a "damaged phase" on the behavior of solids, and to transformation toughening in ceramic composites (Budiansky *et al.*, 1983).

2. PRELIMINARIES

Consider a bar composed of a homogeneous elastic material, which in its reference configuration occupies the interval $[0, L]$. Let x denote the coordinate of a generic point of the bar in this configuration. If the reference cross-sectional area of the bar at x is $A(x) > 0$, it is assumed that $A \in C^2[0, L]$. We also assume that $A(x)$ increases monotonically with x, so that

$$A'(x) > 0, \quad 0 \leqslant x \leqslant L. \tag{1}$$

We shall show later that the special case of the *uniform* bar ($A'(x) = 0$) is exceptional in certain respects; it is temporarily excluded from consideration.

A deformation of the bar is characterized by an invertible mapping

$$y = x+u(x), \quad 0 \leqslant x \leqslant L \tag{2}$$

which subjects the particle at x to a displacement u and carries it to a new location y. There is no loss of generality in taking the left-hand end of the bar to be fixed; if δ denotes the elongation of the bar

$$u(L) = \delta, \quad u(0) = 0. \tag{3}$$

It will be necessary in the following to consider displacement fields which are less than classically smooth, and accordingly we allow for the possibility that, although u is continuous on $[0, L]$, there is a number $s \in [0, L]$ such that (i) u is continuously differentiable on $[0, s]+[s, L]$, (ii) u is twice continuously differentiable on $(0, s)+(s, L)$, and (iii) u' suffers a finite jump discontinuity across $x = s$. The strain ε at a particle $x \neq s$ is defined by

$$\varepsilon(x) = u'(x) > -1, \quad 0 \leqslant x \leqslant L, \quad x \neq s; \tag{4}$$

inequality $(4)_1$ assures the invertibility of mapping (2).

Let $\sigma(x)$ be the nominal stress field in the bar. Equilibrium in the absence of body forces requires

$$\sigma(x)A(x) = F = \text{constant}, \quad 0 \leqslant x \leqslant L; \tag{5}$$

F denotes the force in the bar. Clearly, $\sigma \in C^2[0, L]$.

The material is characterized by an elastic potential W the value of which is the strain energy per unit reference volume. We assume that W is defined on $(-1, \infty)$ and that it is twice continuously differentiable there. The stress response function of the bar $\hat{\sigma}(\varepsilon)$ is given in terms of W by

$$\hat{\sigma}(\varepsilon) = W'(\varepsilon), \quad -1 < \varepsilon < \infty \tag{6}$$

so that by (5), the stress at x is

$$\hat{\sigma}(\varepsilon(x)) = F/A(x), \quad 0 \leqslant x \leqslant L, \quad x \neq s. \tag{7}$$

If F is given, the *force problem* consists of finding a displacement field u of the requisite smoothness conforming to (7), (4) and $(3)_2$. If δ is given, the *elongation problem* requires the determination of a constant F and a displacement field u satisfying (7), (4) and (3). We shall be concerned only with the force problem.

From a thermodynamic viewpoint, the present analysis assumes that conditions are isothermal. The elastic potential W coincides with the Helmholtz free energy of the material at the given temperature, while the associated Gibbs free energy G expressed in terms of strain is

$$G(\varepsilon) = W(\varepsilon)-\hat{\sigma}(\varepsilon)\varepsilon, \quad -1 < \varepsilon < \infty. \tag{8}$$

In this paper we restrict attention to materials the stress response function $\hat{\sigma}(\varepsilon)$ of which first increases with increasing ε, then decreases, and finally increases again (Fig. 1). Specifically we suppose that there are positive numbers ε_M and ε_m such that $\hat{\sigma}'(\varepsilon_M) = \hat{\sigma}'(\varepsilon_m) = 0$, $\hat{\sigma}'(\varepsilon) > 0$ for $-1 < \varepsilon < \varepsilon_M$, $\hat{\sigma}'(\varepsilon) < 0$ for $\varepsilon_M < \varepsilon < \varepsilon_m$, and $\hat{\sigma}'(\varepsilon) > 0$ for $\varepsilon_m < \varepsilon$. Moreover

$$\sigma_M = \hat{\sigma}(\varepsilon_M) > 0, \quad \sigma_m = \hat{\sigma}(\varepsilon_m) > 0, \quad \hat{\sigma}(\infty) = \infty, \quad \hat{\sigma}(-1) = -\infty. \tag{9}$$

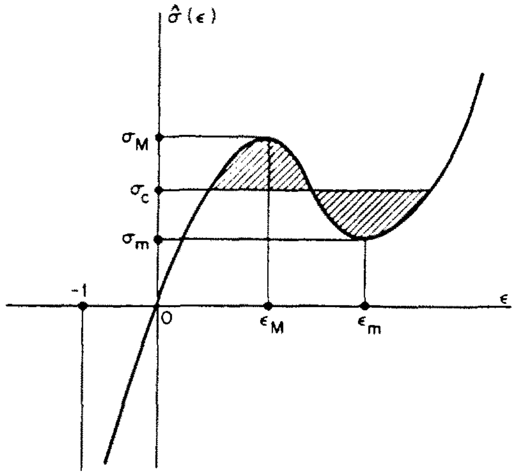


Fig. 1. Stress–strain curve.

Note that these materials are of "Baker–Ericksen type" in the sense that $\hat{\sigma}(\varepsilon)\varepsilon > 0$ for $\varepsilon \neq 0$. For our purposes, it is sufficient to consider only tensile stresses, so we restrict attention henceforth to $\sigma \geqslant 0$.

Although $\hat{\sigma}(\varepsilon)$ is not invertible on $(-1, \infty)$, its restrictions to certain subsets of this interval do have inverses, and these play a major role in the analysis to follow. Let $\hat{\varepsilon}_1$, $\hat{\varepsilon}_2$, $\hat{\varepsilon}_3$ be the functions inverse to the restrictions of $\hat{\sigma}(\varepsilon)$ to the respective intervals $(-1, \varepsilon_M]$, $[\varepsilon_M, \varepsilon_m]$, and $[\varepsilon_m, \infty)$; these inverse functions are defined on $(-\infty, \sigma_M]$, $[\sigma_m, \sigma_M]$ and $[\sigma_m, \infty)$, respectively. Each function $\hat{\varepsilon}_i$ is continuous on its domain of definition, and is continuously differentiable on the interior of that domain. Finally, let σ_c be the unique number in the interval (σ_m, σ_M) for which the two shaded regions in Fig. 1 have equal areas. In terms of the Gibbs free energy

$$G(\hat{\varepsilon}_3(\sigma_c)) = G(\hat{\varepsilon}_1(\sigma_c)); \tag{10}$$

σ_c is the *Maxwell stress* of the material.

3. EQUILIBRIUM STATES

If $\varepsilon(x)$ is a solution of (7) of the requisite smoothness, it follows with the help of inequality (1) that $\varepsilon(x) \neq \varepsilon_m, \varepsilon_M$ for all x in $(0, s) + (s, L)$, and that in fact

$$\varepsilon(x) = \begin{cases} \hat{\varepsilon}_p(F/A(x)), & 0 \leqslant x < s \\ \hat{\varepsilon}_q(F/A(x)), & s < x \leqslant L \end{cases} \tag{11}$$

where $p, q = 1, 2$, or 3. Moreover, for $0 \leqslant x < s$, $F/A(x)$ must lie in the domain of $\hat{\varepsilon}_p$, while for $s < x \leqslant L$ it must lie in the domain of $\hat{\varepsilon}_q$. On using inequality (1) and the definition of the inverse functions $\hat{\varepsilon}_i$, one can show that this is equivalent to requiring

$$(s, F) \in \mathfrak{s}_{pq} \tag{12}$$

where the $\mathfrak{s}_{pq}$'s are sets in the (s, F)-plane defined as follows:

$$\mathfrak{s}_{11} = \{(s, F) \mid 0 \leqslant s \leqslant L, \quad F \leqslant \sigma_M A_m\}$$
$$\mathfrak{s}_{22} = \{(s, F) \mid 0 \leqslant s \leqslant L, \quad \sigma_m A_M \leqslant F \leqslant \sigma_M A_m\}$$
$$\mathfrak{s}_{33} = \{(s, F) \mid 0 \leqslant s \leqslant L, \quad \sigma_m A_M \leqslant F\}$$

$$
\begin{aligned}
\mathfrak{s}_{12} &= \{(s,F)\,|\,0 < s < L,\ \ \sigma_m A_M \leqslant F \leqslant \sigma_M A_m\} \\
\mathfrak{s}_{21} &= \{(s,F)\,|\,0 < s < L,\ \ \sigma_m A(s) \leqslant F \leqslant \sigma_M A_m\} \\
\mathfrak{s}_{23} &= \{(s,F)\,|\,0 < s < L,\ \ \sigma_m A_M \leqslant F \leqslant \sigma_M A_m\} \\
\mathfrak{s}_{32} &= \{(s,F)\,|\,0 < s < L,\ \ \sigma_m A_M \leqslant F \leqslant \sigma_M A(s)\} \\
\mathfrak{s}_{31} &= \{(s,F)\,|\,0 < s < L,\ \ \sigma_m A(s) \leqslant F \leqslant \sigma_M A(s)\} \\
\mathfrak{s}_{13} &= \{(s,F)\,|\,0 < s < L,\ \ \sigma_m A_M \leqslant F \leqslant \sigma_M A_m\}.
\end{aligned}
\tag{13}
$$

Conversely, if $(s, F) \in \mathfrak{s}_{pq}$ for some p, q, then expression (11) is a solution of eqn (7). Observe that the sets $\mathfrak{s}_{22}$, $\mathfrak{s}_{12}$, $\mathfrak{s}_{23}$, $\mathfrak{s}_{13}$ are nonempty if and only if the constitutive law and the taper of the bar are such that

$$\sigma_m A_M \leqslant \sigma_M A_m. \tag{14}$$

For a given material, inequality (14) will certainly be valid if the taper in the bar is slight enough; we assume throughout that inequality (14) holds.

For a given F, all solutions of the force problem (7), (4), $(3)_2$ may now be found by integrating (11). They are

$$u(x) = U_{pq}(x;F,s), \quad 0 \leqslant x \leqslant L, \quad (s,F) \in \mathfrak{s}_{pq}, \quad p,q = 1,2,3 \tag{15}$$

where

$$
U_{pq}(x;F,s) = \begin{cases} \displaystyle\int_0^x \hat{\varepsilon}_p(F/A(\xi))\,\mathrm{d}\xi, & 0 \leqslant x < s \\[2ex] \displaystyle\int_0^s \hat{\varepsilon}_p(F/A(\xi))\,\mathrm{d}\xi + \int_s^x \hat{\varepsilon}_q(F/A(\xi))\,\mathrm{d}\xi, & s < x \leqslant L. \end{cases}
\tag{16}
$$

For $p = q$, (15) and (16) yield the special solutions

$$u(x) = U_p(x;F) \equiv U_{pp}(x;F,s) = \int_0^x \hat{\varepsilon}_p(F/A(\xi))\,\mathrm{d}\xi \tag{17}$$

which are independent of s and classically smooth. On the other hand, for $p \neq q$, (15) and (16) provide six one-parameter families of solution to the force problem, with parameter s. The strain for $0 \leqslant x < s$ is associated with the pth branch of the stress–strain curve, while that for $s < x \leqslant L$ is associated with the qth branch; the discontinuity at $x = s$ is called a "(p, q)-shock".

According to (13) and (14), there exists at least one solution $u(x)$ to the force problem corresponding to every given value of F. If either $0 \leqslant F \leqslant \sigma_m A_m$ or $F \leqslant \sigma_M A_M$, this solution is unique and it is smooth. However, for values of force in the intermediate range $\sigma_m A_m < F < \sigma_M A_M$, there are infinitely many solutions.

Observe from (16) and (17) that as the shock recedes to either one of the two ends of the bar, each weak solution "merges" with one of the smooth solutions

$$
\left.\begin{aligned}
\lim_{s \to 0} U_{pq}(x;F,s) &= U_q(x;F) \\
\lim_{s \to L} U_{pq}(x;F,s) &= U_p(x;F).
\end{aligned}\right\}
\tag{18}
$$

This suggests that the definitions of all of the U_{pq}'s, as functions of s, be extended to $s = 0$ and L by setting

$$
\left.\begin{aligned}
U_{pq}(x;F,0) &= U_q(x;F) \\
U_{pq}(x;F,L) &= U_p(x;F).
\end{aligned}\right\}
\tag{19}
$$

Finally, one can verify that $U_{pq}(\cdot\,;F,s_1)$ and $U_{mn}(\cdot\,;F,s_2)$ are distinct whenever $(p,q) \neq (m,n)$, $0 < s_1 < L$, and $0 < s_2 < L$:

$$U_{pq}(\cdot\,;F,s_1) \neq U_{mn}(\cdot\,;F,s_2) \quad \text{if} \quad (p,q) \neq (m,n), \quad 0 < s_1 < L, \quad 0 < s_2 < L,$$
$$(s_1,F) \in \mathfrak{s}_{pq}, \quad (s_2,F) \in \mathfrak{s}_{mn}. \tag{20}$$

We turn next to the relation between the force F and the elongation δ, which we call the *macroscopic response* of the bar. By $(3)_1$, (15) and (16), these quantities are related by

$$\delta = \Delta_{pq}(F,s), \quad (s,F) \in \mathfrak{s}_{pq}, \quad p,q = 1,2,3 \tag{21}$$

where

$$\Delta_{pq}(F,s) = U_{pq}(L;F,s), \quad (s,F) \in \mathfrak{s}_{pq}, \quad p,q = 1,2,3. \tag{22}$$

It can be shown that, if $p \neq q$, $\Delta_{pq}(F,s)$ is a monotonic function of s for each fixed F. The macroscopic response corresponding to any one of the smooth solutions is independent of s:

$$\delta = \Delta_{pp}(F,s) \equiv \Delta_p(F). \tag{23}$$

For each (p,q), (21) maps the set $\mathfrak{s}_{pq}$ of the (s,F)-plane onto a set $\mathfrak{E}_{pq}$ in the (δ,F)-plane:

$$\begin{aligned}
\mathfrak{E}_{11} &= \{(\delta,F) \mid \delta = \Delta_1(F), \quad F \leqslant \sigma_M A_m\} \\
\mathfrak{E}_{22} &= \{(\delta,F) \mid \delta = \Delta_2(F), \quad \sigma_m A_M \leqslant F \leqslant \sigma_M A_m\} \\
\mathfrak{E}_{33} &= \{(\delta,F) \mid \delta = \Delta_3(F), \quad \sigma_m A_M \leqslant F\} \\
\mathfrak{E}_{12} &= \{(\delta,F) \mid \Delta_{12}(F,L) \leqslant \delta \leqslant \Delta_{12}(F,0), \quad \sigma_m A_M \leqslant F \leqslant \sigma_M A_m\} \\
\mathfrak{E}_{21} &= \{(\delta,F) \mid \Delta_{21}(F,0) \leqslant \delta \leqslant \Delta_{21}(F,L), \quad \sigma_m A(s) \leqslant F \leqslant \sigma_M A_m\} \\
\mathfrak{E}_{23} &= \{(\delta,F) \mid \Delta_{23}(F,L) \leqslant \delta \leqslant \Delta_{23}(F,0), \quad \sigma_m A_M \leqslant F \leqslant \sigma_M A_m\} \\
\mathfrak{E}_{32} &= \{(\delta,F) \mid \Delta_{32}(F,0) \leqslant \delta \leqslant \Delta_{32}(F,L), \quad \sigma_m A_M \leqslant F \leqslant \sigma_M A(s)\} \\
\mathfrak{E}_{31} &= \{(\delta,F) \mid \Delta_{31}(F,0) \leqslant \delta \leqslant \Delta_{31}(F,L), \quad \sigma_m A(s) \leqslant F \leqslant \sigma_M A(s)\} \\
\mathfrak{E}_{13} &= \{(\delta,F) \mid \Delta_{13}(F,L) \leqslant \delta \leqslant \Delta_{13}(F,0), \quad \sigma_m A_M \leqslant F \leqslant \sigma_M A_m\}.
\end{aligned} \tag{24}$$

Sketches of the sets $\mathfrak{E}_{pq}$ are shown in Fig. 2. Observe that $\mathfrak{E}_{11}$ and $\mathfrak{E}_{33}$ are curves with positive slope, while $\mathfrak{E}_{22}$ is a curve with negative slope. Note also that $\mathfrak{E}_{11}$, $\mathfrak{E}_{22}$ and $\mathfrak{E}_{33}$ are

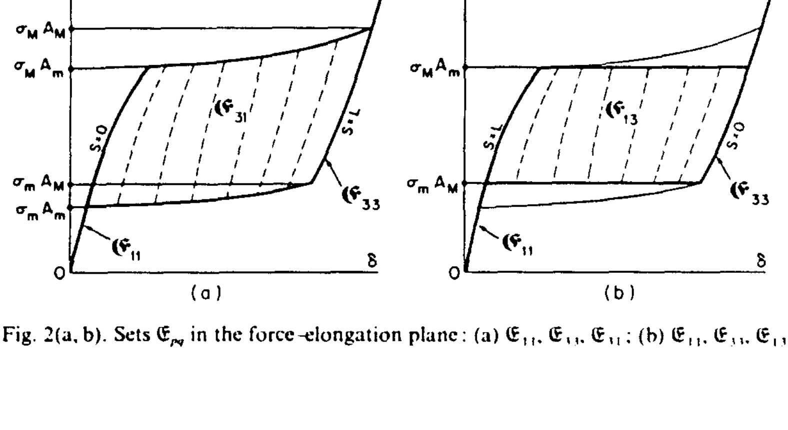


Fig. 2(a, b). Sets $\mathfrak{E}_{pq}$ in the force–elongation plane: (a) $\mathfrak{E}_{11}$, $\mathfrak{E}_{33}$, $\mathfrak{E}_{31}$; (b) $\mathfrak{E}_{11}$, $\mathfrak{E}_{33}$, $\mathfrak{E}_{13}$.

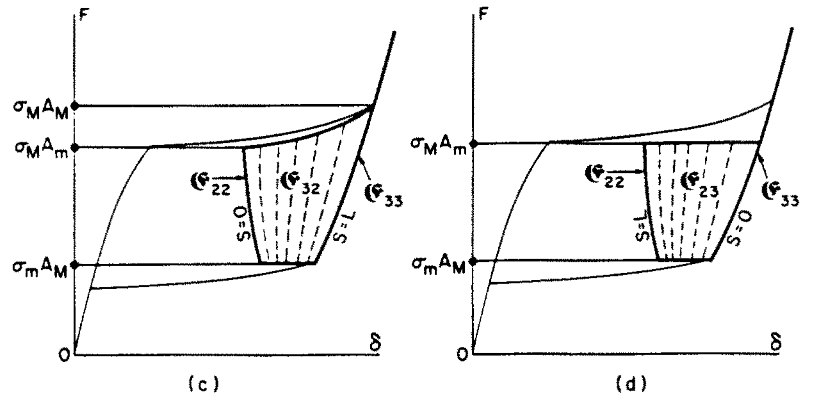


Fig. 2(c, d). Sets $\mathfrak{E}_{pq}$ in the force–elongation plane: (c) $\mathfrak{E}_{22}$, $\mathfrak{E}_{33}$, $\mathfrak{E}_{32}$; (d) $\mathfrak{E}_{22}$, $\mathfrak{E}_{33}$, $\mathfrak{E}_{23}$.

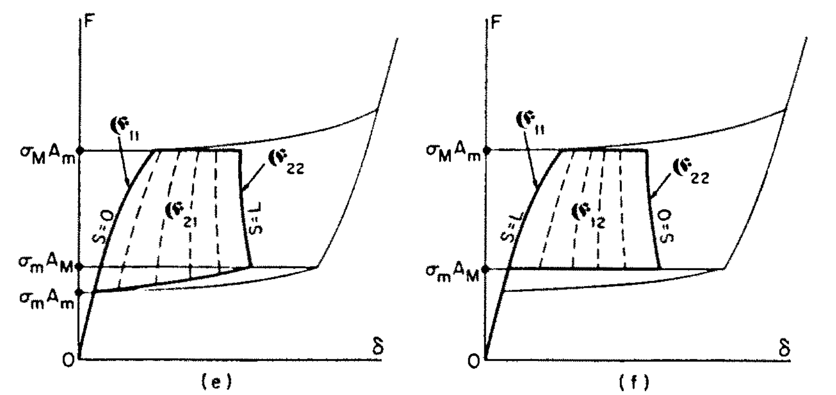


Fig. 2(c, f). Sets $\mathfrak{E}_{pq}$ in the force–elongation plane: (e) $\mathfrak{E}_{11}$, $\mathfrak{E}_{22}$, $\mathfrak{E}_{21}$; (f) $\mathfrak{E}_{11}$, $\mathfrak{E}_{22}$, $\mathfrak{E}_{12}$.

not connected. The sets $\mathfrak{E}_{pq}$, $p \neq q$, correspond to various regions linking these curves. The dashed curves in these regions are curves of constant s. For $p \neq q$, the mappings $\mathfrak{s}_{pq} \rightarrow \mathfrak{E}_{pq}$ are one-to-one; this is obviously not the case when $p = q$.

In summary, for sufficiently small and sufficiently large values of the force F, the force problem has a unique solution; this solution happens to be smooth. However, for intermediate values of F, we encounter a major breakdown of uniqueness. In fact, in the intermediate range of F, there are multiple solutions even if the *pair* (δ, F) is prescribed.

4. DISSIPATION, SHOCK DRIVING TRACTION, ADMISSIBILITY

We now turn our attention to quasi-static motions of the bar in which, at each instant t, the displacement field $u(\cdot, t)$ corresponds to one of the equilibrium states constructed in the preceding section. Let $F(t)$, $t_0 \leqslant t \leqslant t_1$, be a given continuous, piecewise continuously differentiable force history. Suppose first that $F(t) \leqslant \sigma_m A_m$ for all t in $[t_0, t_1]$. Then by (13) and (17), $u(x, t)$ is necessarily given by the smooth field

$$u(x,t) = U_1(x; F(t)), \quad 0 \leqslant x \leqslant L, \quad t_0 \leqslant t \leqslant t_1 \tag{25}$$

associated with the first branch of the stress–strain curve. Next, suppose that $F(t) \geqslant \sigma_M A_M$ for all t. Then (13) and (17) yield

$$u(x,t) = U_3(x; F(t)), \quad 0 \leqslant x \leqslant L, \quad t_0 \leqslant t \leqslant t_1 \tag{26}$$

and again the field is smooth. Finally, assume that $\sigma_m A_m \leqslant F(t) \leqslant \sigma_M A_M$ for all t in $[t_0, t_1]$.

Then $u(x,t)$ must have the form

$$u(x,t) = U_{p(t)q(t)}(x;F(t),s(t)), \quad 0 \leqslant x \leqslant L, \quad t_0 \leqslant t \leqslant t_1. \tag{27}$$

To begin with, we assume that:

(i) $p(t)$ and $q(t)$ are piecewise constant on $[t_0,t_1]$, each taking one of the values 1, 2, 3 there, with $p(t) \neq q(t)$;

(ii) $s(t)$ is piecewise continuous on $[t_0,t_1]$;

(iii) $(s(t),F(t)) \in \mathfrak{s}_{p(t)q(t)}$ for $t_0 \leqslant t \leqslant t_1$.

The requirement $p(t) \neq q(t)$ does not preclude the occurrence of smooth fields for forces $F(t)$ in this intermediate range; such fields occur when either $s(t) = 0$ or L.

For quasi-static motions of the form (25) or (26), the assumed smoothness of $F(t)$ and representations (17) guarantee that $u(x,\cdot)$ is continuous and piecewise continuously differentiable on $[t_0,t_1]$ for each x. In order to discuss the more complicated issue of the smoothness in time of motions described by (27), it is convenient first to introduce the notion of a *transition instant*: an instant $t_* \in (t_0,t_1)$ is a transition instant if $(p(t_*-), q(t_*-)) \neq (p(t_*+),q(t_*+))$. At a transition instant, the branches of the underlying stress–strain curve involved in deformation (27) change from the $p(t-)$th and $q(t-)$th to the $p(t+)$th and $q(t+)$th.

It is natural to require that $u(x,\cdot)$ as given by (27) *also* be continuous on $[t_0,t_1]$ for every x, $0 \leqslant x \leqslant L$. Let t_* be in (t_0,t_1), and assume that $u(x,\cdot)$ is continuous at t_*. Suppose first that t_* is *not* a transition instant. Then $p(t-) = p(t+) \equiv p$, $q(t-) = q(t+) \equiv q$, so from (27)

$$u(x,t_*+) - u(x,t_*-) = U_{pq}(x;F(t_*),s(t_*+)) - U_{pq}(x;F(t_*),s(t_*-)) = 0 \tag{28}$$

for every x in $[0,L]$. Now $p \neq q$, and definition (16) of U_{pq} shows that (28) cannot hold under this circumstance unless

$$s(t_*+) = s(t_*-) \quad \text{if } t_* \text{ is not a transition instant.} \tag{29}$$

Thus $s(t)$ is necessarily continuous at all times except possibly at transition instants. Now suppose that t_* *is* a transition instant. Let $p(t_*-) \equiv p$, $q(t_*-) \equiv q$, $p(t_*+) \equiv m$, $q(t_*+) \equiv n$; by (27), continuity of $u(x,\cdot)$ at $t = t_*$ then implies that

$$U_{pq}(x;F(t_*),s(t_*+)) - U_{mn}(x;F(t_*),s(t_*-)) = 0 \tag{30}$$

for all x in $[0,L]$. Since t_* is a transition instant, $(p,q) \neq (m,n)$, and (20) shows that (30) cannot hold unless at least one of the numbers $s(t_*+)$, $s(t_*-)$ takes either the value 0 or the value L. More detailed examination of (30) shows that one of the following four mutually exclusive possibilities must hold:

$$\left.\begin{array}{ll} s(t_*+) = s(t_*-) = 0, \text{ and } q(t_*+) = q(t_*-) & (31a) \\ s(t_*+) = s(t_*-) = L, \text{ and } p(t_*+) = p(t_*-) & (31b) \\ s(t_*+) = L,\ s(t_*-) = 0, \text{ and } p(t_*+) = q(t_*-) & (31c) \\ s(t_*+) = 0,\ s(t_*-) = L, \text{ and } p(t_*-) = q(t_*+) & (31d) \end{array}\right\} \text{if } t_* \text{ is a transition instant.}$$

Thus (29) and (31) are necessary for the continuity of $u(x,\cdot)$ at an instant t_* in (t_0,t_1); if either $t_* = t_0$ or $t_* = t_1$, (29) and (31) continue to be necessary, provided the appropriate $+$ or $-$ is deleted in the arguments of s, p and q. Moreover, one can show that, in the presence of the assumed smoothness of $F(t)$, (29) and (31) (or their modified versions when t_* is an end-point of $[t_0,t_1]$) are *sufficient* for the continuity of $u(x,\cdot)$ at $t = t_*$ as well.

The argument above shows that discontinuities in $s(t)$ can only occur at transition instants t_*; if there is such a discontinuity, the shock $x = s(t)$ recedes to one end of the bar as $t \to t_*-$ and then advances into the bar from the other end as t increases from t_*. When $s(t)$ is continuous at a transition instant t_*, necessarily $s(t_*) = 0$ or L. Thus a transition from a discontinuous strain field involving branches p and q of the stress–strain curve to one involving branches m and n, with $(p, q) \neq (m, n)$, *always takes place through a smooth field*. A further consequence of conditions (29) and (31) is that a shock cannot emerge instantaneously from a smooth field at an interior point of the bar. Observe that these restrictions on the motion $x = s(t)$ of the shock arise from purely kinematic requirements, together with the assumption that the bar is strictly monotonically tapered. Further restrictions on the shock motion will arise later.

Finally, we require that $s(t)$ should be piecewise continuously differentiable between every pair of successive transition instants. This will assure that $u(x, \cdot)$ is piecewise continuously differentiable on $[t_0, t_1]$. A *regular instant* during a quasi-static motion is a time t at which $\dot{F}(t)$ exists and, if the motion is of the form (27), $\dot{s}(t)$ exists and $p(t)$ and $q(t)$ are continuous.

Figure 3 describes an example in the (x, t)-plane for which the shock history involves transition instants of each of the four kinds listed in (31a)–(31d). The encircled numbers in Fig. 3 refer to the branches of the stress–strain curve appropriate to the two sides of the shock at various times.

The elementary quasi-static motions (25), (26) and (27) may be linked together on successive time intervals $[t_0, t_1]$, $[t_1, t_2]$, and $[t_2, t_3]$ to form a compound quasi-static motion on $[t_0, t_3]$, provided the resulting displacment $u(x, \cdot)$ is continuous on $[t_0, t_3]$ for every x.

Next we consider the *energetics* of a quasi-static motion. The total strain energy stored in the bar in an equilibrium state with displacement $U_{pq}(x; F, s)$ is

$$E_{pq}(F, s) = \int_0^s W(\hat{\varepsilon}_p(F/A(x)))A(x)\,\mathrm{d}x + \int_s^L W(\hat{\varepsilon}_q(F/A(x)))A(x)\,\mathrm{d}x,$$

$$p, q = 1, 2, 3, \quad (s, F) \in \mathfrak{s}_{pq}. \quad (32)$$

During a quasi-static motion of the form (27), the energy stored in the bar at time t is

$$E(t) = E_{p(t)q(t)}(F(t), s(t)), \quad t_0 \leqslant t \leqslant t_1. \quad (33)$$

At a regular instant during this motion, we define the *rate of dissipation* $d(t)$ to be the difference between the rate of external work and the rate of increase of stored energy

$$d(t) = F(t)\dot{\delta}(t) - \dot{E}(t) \quad (34)$$

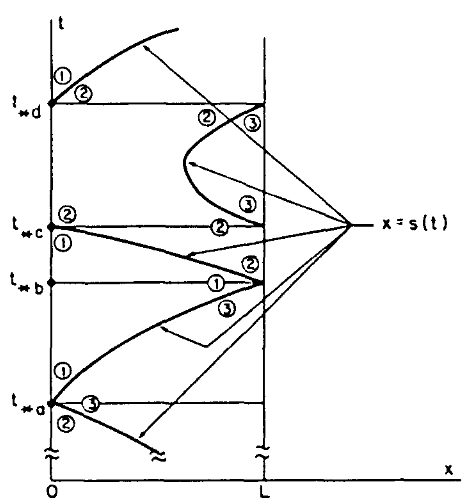


Fig. 3. Examples of the four types of transition instants t_* in (31).

where, by (21), the elongation $\delta(t)$ of the bar is

$$\delta(t) = \Delta_{p(t)q(t)}(F(t), s(t)), \quad t_0 \leqslant t \leqslant t_1 \tag{35}$$

and Δ_{pq} is given by (22). A direct calculation based on (32)–(35) yields

$$d(t) = [[W(\varepsilon) - \hat{\sigma}(\varepsilon)\varepsilon]]^{+}_{-} A(s(t))\dot{s}(t) \tag{36}$$

for each regular instant during the motion. Here $[[g]]$ denotes the jump at $x = s(t)$ of function $g(x)$: $[[g]] = g(s(t)+) - g(s(t)-)$. It follows from (36) that $d(t) = 0$ if the motion happens to be smooth at time t, so that all jumps in (36) vanish. In general, however, $d(t) \neq 0$ for motions of the type (27). For motions of the type (25) or (26), the dissipation rate of course vanishes identically.

Let $f(t)$ be defined by

$$f(t) = [[W(\varepsilon) - \hat{\sigma}(\varepsilon)\varepsilon]]^{+}_{-}, \quad t_0 \leqslant t \leqslant t_1 \tag{37}$$

and by $f(t) = 0$ for motions of the type (25) or (26). Since by (34), (36) and (37), at a regular instant

$$\dot{E} = F\dot{\delta} + (-fA(s))\dot{s} \tag{38}$$

it follows that one can view $-f(t)$ as a traction applied by the shock on the bar, or equivalently, $f(t)$ as a traction applied by the bar on the shock. The general notion of a "thermodynamic force" or "driving force" on a "defect" was introduced by Eshelby (1956). Equation (37) is a special case of a formula for the force on a phase boundary given by Eshelby (1970) and discussed by Rice (1975); see also Knowles (1979). We refer to f as the "*shock driving traction*". Observe from (8) and (36), that the shock driving traction coincides with the jump in the Gibbs free energy across the shock

$$f(t) = [[G(\varepsilon(x, t))]]^{+}_{-}. \tag{39}$$

On using (10), one obtains from (39) the following expression for the shock driving traction:

$$f(t) = f_{p(t)q(t)}(\bar{\sigma}(t)) \tag{40}$$

where $\bar{\sigma}(t)$ is the nominal stress at the shock

$$\bar{\sigma}(t) = F(t)/A(s(t)) \tag{41}$$

and functions f_{pq} are determined by the material; they are given by

$$f_{pq}(\sigma) = \int_{\hat{\varepsilon}_p(\sigma)}^{\hat{\varepsilon}_q(\sigma)} \hat{\sigma}(\varepsilon)\, d\varepsilon - \sigma\{\hat{\varepsilon}_q(\sigma) - \hat{\varepsilon}_p(\sigma)\}, \quad \sigma_m \leqslant \sigma \leqslant \sigma_M, \quad p, q = 1, 2, 3. \tag{42}$$

The following properties of functions f_{pq} will prove to be useful. First, note from (42) that

$$f_{pq}(\sigma) = -f_{qp}(\sigma), \quad \sigma_m \leqslant \sigma \leqslant \sigma_M, \quad p, q = 1, 2, 3. \tag{43}$$

Second, differentiating (42) yields

$$f'_{pq}(\sigma) = \hat{\varepsilon}_p(\sigma) - \hat{\varepsilon}_q(\sigma) \begin{cases} > 0, & \sigma_m \leqslant \sigma \leqslant \sigma_M, \quad p > q \\ = 0, & \sigma_m \leqslant \sigma \leqslant \sigma_M, \quad p = q \\ < 0, & \sigma_m \leqslant \sigma \leqslant \sigma_M, \quad p < q. \end{cases} \tag{44}$$

Next, on recalling the properties of the inverse functions ε_i, one can readily verify from (42) that

$$\left.\begin{array}{lll} f_{31}(\sigma_m) < 0, & f_{31}(\sigma_c) = 0, & f_{31}(\sigma_M) > 0 \\ f_{21}(\sigma_M) = 0, & f_{32}(\sigma_m) = 0. & \end{array}\right\} \tag{45}$$

Integration of (44) with the help of (45) then gives the following useful alternative formulas for f_{pq}:

$$\left.\begin{array}{ll} f_{32}(\sigma) = \displaystyle\int_{\sigma_m}^{\sigma} \{\hat{\varepsilon}_3(\tau) - \hat{\varepsilon}_2(\tau)\}\, d\tau, & \sigma_m \leqslant \sigma \leqslant \sigma_M \\ f_{31}(\sigma) = \displaystyle\int_{\sigma_c}^{\sigma} \{\hat{\varepsilon}_3(\tau) - \hat{\varepsilon}_1(\tau)\}\, d\tau, & \sigma_m \leqslant \sigma \leqslant \sigma_M \\ f_{21}(\sigma) = -\displaystyle\int_{\sigma}^{\sigma_M} \{\hat{\varepsilon}_2(\tau) - \hat{\varepsilon}_1(\tau)\}\, d\tau, & \sigma_m \leqslant \sigma \leqslant \sigma_M; \end{array}\right\} \tag{46}$$

the other f_{pq}'s are related to these through (43). Observe from (46) that f_{31}, f_{32} and f_{21} are monotonically increasing functions on (σ_m, σ_M). Moreover, f_{32} is nonnegative on its domain of definition while f_{21} is nonpositive; on the other hand f_{31} changes sign once:

$$\left.\begin{array}{l} f_{32}(\sigma)\begin{cases} > 0, & \sigma_m < \sigma \leqslant \sigma_M \\ = 0, & \sigma = \sigma_m \end{cases} \\ f_{31}(\sigma)\begin{cases} > 0, & \sigma_c < \sigma \leqslant \sigma_M \\ = 0, & \sigma = \sigma_c \\ < 0, & \sigma_m \leqslant \sigma < \sigma_c \end{cases} \\ f_{21}(\sigma)\begin{cases} = 0, & \sigma = \sigma_M \\ < 0, & \sigma_m \leqslant \sigma < \sigma_M. \end{cases} \end{array}\right\} \tag{47}$$

A quasi-static motion is said to be *admissible* if the rate of dissipation is nonnegative

$$f(t)\dot{s}(t+) \geqslant 0, \quad t_0 \leqslant t \leqslant t_1. \tag{48}$$

On adapting the thermodynamic arguments given by Knowles (1979) to the present one-dimensional context, one can show that (48) is equivalent to the Clausius–Duhem inequality when the temperature is constant in space and time. Observe that (48) holds with equality if the shock is stationary or if the shock driving traction vanishes; the latter alternative occurs if and only if either (i) the field is smooth or (ii) $(p, q) = (3, 1)$ or $(1, 3)$ and the stress at the shock is the Maxwell stress. All motions of the type (45) or (47) are trivially admissible. In general, (48) is to be viewed as a restriction on allowable quasi-static motions. Note from (35) and (48) that a quasi-static motion is admissible at an instant t if and only if the Gibbs free energy of a particle at the shock front does not increase as the particle crosses the shock. In the following section, we investigate the consequences of admissibility.

5. CONSEQUENCES OF ADMISSIBILITY

Consider an admissible quasi-static motion of the form (27) in which $p(t) \equiv p =$ constant, $q(t) \equiv q =$ constant for $t_0 \leqslant t \leqslant t_1$. If, for example, $p = 3$, $q = 1$, then by (44), (36) and (43), admissibility requires that

$$\left.\begin{array}{rl} \dot{s}(t+) \geqslant 0 & \text{if} \quad \bar{\sigma}(t) > \sigma_c \\ \dot{s}(t+) \leqslant 0 & \text{if} \quad \bar{\sigma}(t) < \sigma_c \\ \dot{\bar{\sigma}}(t+)\dot{s}(t+) \geqslant 0 & \text{if} \quad \bar{\sigma}(t) = \sigma_c \end{array}\right\} \tag{49}$$

where $\bar{\sigma}(t) \equiv F(t)/A(s(t))$ is the stress at the shock. The curve in $\mathfrak{E}_{31}$ given by $F = \sigma_c A(s)$, $\delta = \Delta_{31}(\sigma_c A(s), s)$ is called the *3.1-Maxwell curve*; points on this curve correspond to equilibrium fields in which the stress at the shock coincides with the Maxwell stress.

Every quasi-static motion determines a path in the (δ, F)-plane. For the motion considered above, this path lies in the set $\mathfrak{E}_{31}$ (Fig. 2), and—through (49)—admissibility restricts the possible directions of the path. Figure 4(a) illustrates the permissible directions of departure of such a path from various points in $\mathfrak{E}_{31}$. The dashed curves in the figure represent lines $s(t)$ = constant; for motions the paths of which lie along a curve s = constant, there is no dissipation. The same is true for motions the paths of which lie on the Maxwell curve.

Similar considerations apply to motions of the type (27) for other values of the pair (p, q). For $p = 1, q = 3$, there is also a Maxwell curve, but for the remaining possible choices of (p, q), this is not the case. Figures 4(b)–(f) show the admissible directions for the remaining choices of (p, q). In Figs 4(a) and (b) and hereafter, we assume that

$$\sigma_m/\sigma_c < A_m/A_M, \quad \sigma_c/\sigma_M < A_m/A_M. \tag{50}$$

These inequalities, which imply (14), certainly hold for a given material if the taper of the bar is slight enough.

Admissible directions for compound motions can also be read off from Fig. 4.

From Fig. 4(a), it is clear that transitions of the form $\mathfrak{E}_{11} \to \mathfrak{E}_{31}$, $\mathfrak{E}_{31} \to \mathfrak{E}_{11}$, $\mathfrak{E}_{33} \to \mathfrak{E}_{31}$, $\mathfrak{E}_{11} \to \mathfrak{E}_{33}$ are all possible; similar inferences may be drawn from Fig. 4(b). The situation is different, however, for admissible motions that involve branch 2 of the stress-strain curve. For example, Fig. 4(c) shows that, while the transition $\mathfrak{E}_{21} \to \mathfrak{E}_{11}$ is admissible, the reverse transition is not; similarly, $\mathfrak{E}_{22} \to \mathfrak{E}_{21}$ is possible, but $\mathfrak{E}_{21} \to \mathfrak{E}_{22}$ is not. Parallel conclusions follow from Figs 4(d)–(f). In general, one can readily show that admissible quasi-static motions proceed in such a way as to diminish—or at least not increase—the length of the portion of the bar bearing strains associated with branch 2 of the stress-strain curve.

The above considerations suggest that the totality of all equilibrium displacement fields be divided into two disjoint parts A and U as follows. Let

$$\begin{aligned} A &= \{U_{pq}(\cdot\,; F, s) \mid (s, F) \in \mathfrak{s}_{pq},\ p = 1 \text{ or } 3, q = 1 \text{ or } 3\} \\ U &= \{U_{pq}(\cdot\,; F, s) \mid (s, F) \in \mathfrak{s}_{pq},\ \text{either } p = 2 \text{ or } q = 2\}. \end{aligned} \tag{51}$$

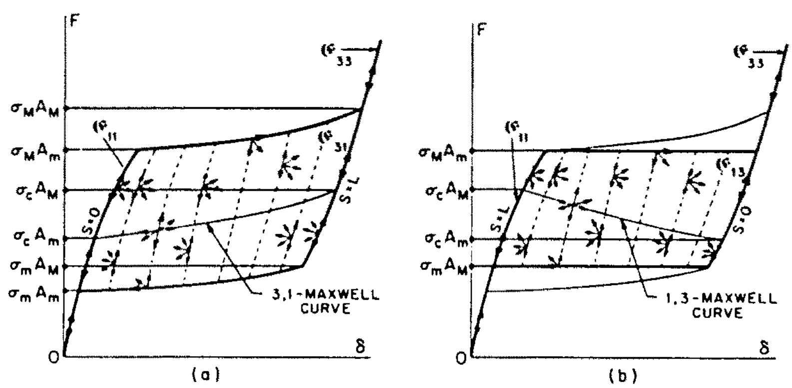


Fig. 4(a, b). Admissible directions in: (a) $\mathfrak{E}_{11}$, $\mathfrak{E}_{33}$, $\mathfrak{E}_{31}$; (b) $\mathfrak{E}_{11}$, $\mathfrak{E}_{33}$, $\mathfrak{E}_{13}$.

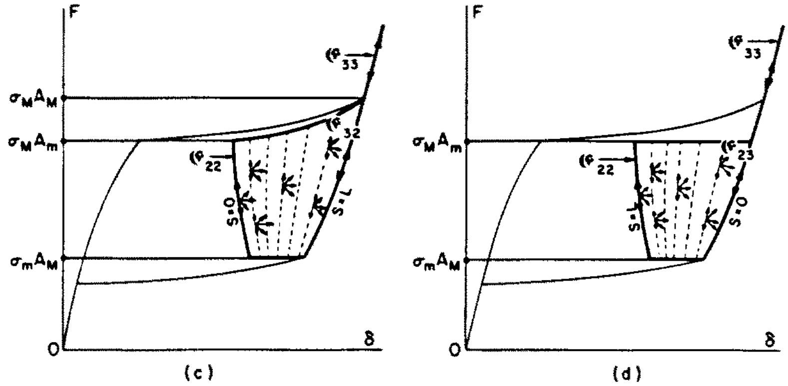


Fig. 4(c, d). Admissible directions in: (c) $\mathfrak{E}_{22}$, $\mathfrak{E}_{33}$, $\mathfrak{E}_{32}$; (d) $\mathfrak{E}_{22}$, $\mathfrak{E}_{33}$, $\mathfrak{E}_{23}$.

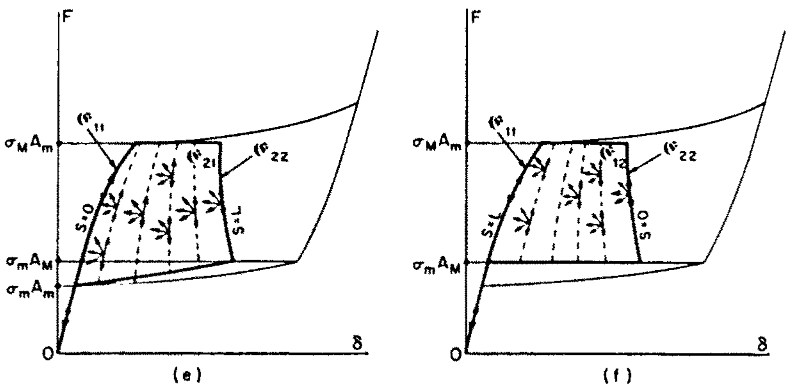


Fig. 4(e, f). Admissible directions in: (e) $\mathfrak{E}_{11}$, $\mathfrak{E}_{22}$, $\mathfrak{E}_{21}$; (f) $\mathfrak{E}_{11}$, $\mathfrak{E}_{22}$, $\mathfrak{E}_{12}$.

Thus the fields in A correspond to those equilibrium states that involve no strains associated with branch 2 of the stress–strain curve in Fig. 1; U contains all remaining displacement fields. Consider a compound admissible motion the displacement field of which at the initial instant belongs to A. Figure 4 suggests that displacement fields for this motion at all later times must also belong to A. Indeed, if at some later instant the corresponding displacement field were in U, the length of that portion of the bar carrying branch 2 strains would necessarily have increased at some earlier time, contradicting the assumed admissibility of the motion. Thus states in U are not accessible from states in A during an admissible quasi-static motion; in particular, a motion that commences at the unloaded, unextended state $F = \delta = 0$ can never enter the collection U of states involving strains on branch 2 of the stress–strain curve.

Let $d(t)$ be the rate of dissipation in a quasi-static motion at each regular instant t; the *total* dissipation D associated with the motion is

$$D = \int_{t_0}^{t_1} d(t)\,\mathrm{d}t. \tag{52}$$

It is possible to show that all admissible quasi-static motions the total dissipation of which is sufficiently large must ultimately enter the collection A, where—in view of the discussion above—they will subsequently remain. We shall not prove this result here (see Abeyaratne and Knowles (1987a, b) for proofs of closely related propositions).

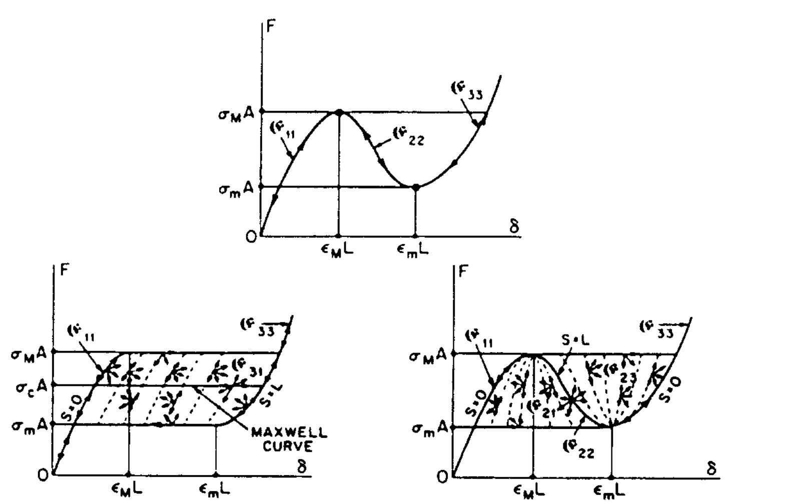
One may thus regard the collection A of equilibrium fields as an *attractor* for admissible quasi-static motions; the set U may be thought of as unobservable. From here on, we shall be concerned only with motions through equilibrium states that can be reached admissibly from the state $F = \delta = 0$; as a result, we need no longer consider states that include strains associated with branch 2 of the stress–strain curve.

In reaching the conclusions described above, the fact that the cross-section of the bar is tapered, rather than uniform is crucial. For a uniform bar, the implications of admissibility are much weaker than those described above. The distinction can be appreciated with the help of Fig. 5, which shows the sets $\mathfrak{E}_{pq}$ for the *uniform* bar. Observe from the figure that the sets $\mathfrak{E}_{11}$, $\mathfrak{E}_{22}$ and $\mathfrak{E}_{33}$ corresponding to *smooth* strain fields are now connected, each to the next, in contrast to the corresponding sets for the tapered bar as shown in Fig. 2 or Fig. 4. Thus for a tapered bar, one cannot move quasi-statically from states in $\mathfrak{E}_{11}$ to states in $\mathfrak{E}_{22}$ by utilizing smooth fields alone; such a transition would demand the presence of fields with equilibrium shocks, which—by the discussion above—is forbidden when admissibility is imposed as a requirement. On the other hand Fig. 5 shows that, for the uniform bar, one can construct quasi-static motions involving only smooth—and therefore trivially admissible—fields that pass from states in $\mathfrak{E}_{11}$ to states in $\mathfrak{E}_{22}$. Thus while admissibility forces the collection U of equilibrium states involving branch 2 to be inaccessible and transient in the presence of sufficient dissipation in the case of the tapered bar, it does not deliver the corresponding results for the uniform bar.

6. KINETIC RELATIONS AND SHOCK INITIATION

The specification of either the force history $F(t)$ or the elongation history $\delta(t)$ during an admissible quasi-static motion is not sufficient to determine the motion uniquely unless the field is smooth at each instant. This suggests that the constitutive characterization of the material must be supplemented if the response is to be determinate when equilibrium shocks are present.

In the macroscopic F–δ relations (21)–(23) pertaining to equilibrium states with shocks, the shock location s may be viewed as an "internal variable". Indeed, the formalism used in internal variable theories of plasticity such as those developed in Rice (1970, 1971, 1975) has a counterpart in the present context. Because we do not need this formalism here, we do not discuss it in detail (it has been described in a related setting in Abeyaratne and Knowles (1987a, b)). A common ingredient of such microstructural theories of inelastic behavior is an evolution law, or "kinetic relation", relating the time rate of change of the internal variable to the corresponding "thermodynamic force". We adopt this point of view here by postulating a relation between the shock driving traction $f(t)$ of (37) and the shock velocity $\dot{s}(t)$ that must hold during a quasi-static motion. This relation is determined by the material and is regarded as given.



FOR $\mathfrak{E}_{13}$, INTERCHANGE S=O AND S=L. FOR $\mathfrak{E}_{12}$ AND $\mathfrak{E}_{32}$, INTERCHANGE S=O AND S=L.

Fig. 5. Sets $\mathfrak{E}_{pq}$ and admissible directions for the uniform bar.

Suppose we have an admissible motion of the form (27). Recall relation (36) between the shock driving traction $f(t)$ during such a motion and the stress at the shock $\bar{\sigma}(t)$, and let

$$M_{pq} = \max_{\sigma_m \leqslant \sigma \leqslant \sigma_M} f_{pq}(\sigma), \quad m_{pq} = \min_{\sigma_m \leqslant \sigma \leqslant \sigma_M} f_{pq}(\sigma) \tag{53}$$

be the maximum and minimum values of the material functions f_{pq} introduced in (42). For each $p, q = 1, 2, 3$ with $p \neq q$, we assume there is a given function V_{pq} defined on $(m_{pq}, M_{pq}) \times [0, L]$ and such that, between successive transition instants during the motion, the *kinetic relation*

$$\dot{s}(t) = V_{p(t)q(t)}(f(t), s(t)) \tag{54}$$

holds. If the functions $V_{pq}(f, s)$ are independent of s, we say that the kinetic relation is *homogeneous*; we assume this to be the case henceforth.

We now impose on each kinetic response function V_{pq} the requirement that

$$V_{pq}(f)f \geqslant 0, \quad m_{pq} < f < M_{pq}, \quad 0 \leqslant s \leqslant L; \tag{55}$$

by (48) and (54), this assures that all quasi-static motions compatible with the kinetic relation are admissible.

We also require that, for each p, q with $p \neq q$

$$V_{qp}(f) = -V_{pq}(-f) \quad \text{for} \quad -M_{pq} < f < -m_{pq}. \tag{56}$$

(Note that, by (33) and (53), $m_{qp} = -M_{pq}$.) The motivation for (56) lies in the fact that the kinetic response functions V_{pq} are to depend only on the material and not on the geometry of the bar under consideration. In particular, they must apply in the case of uniform bars, for which (3, 1)-shocks and (1, 3)-shocks are mirror-images of one another if the stress at the shock is the same for both.

Since the shock driving traction $f(t)$ during the motion is related to the stress at the shock $\bar{\sigma}(t)$ by (40), the kinetic relation (54) may be expressed in terms of $\bar{\sigma}$ instead of f:

$$\dot{s}(t) = v_{p(t)q(t)}(\bar{\sigma}(t)) = v_{p(t)q(t)}(F(t)/A(s(t))) \tag{57}$$

where the material function v_{pq} is defined by

$$v_{pq}(\sigma) = V_{pq}(f_{pq}(\sigma)), \quad \sigma_m < \sigma < \sigma_M. \tag{58}$$

By (40) and (43), properties (55) and (56) of the V_{pq}'s yield corresponding properties of the v_{pq}'s

$$v_{pq}(\sigma)f_{pq}(\sigma) \geqslant 0, \quad \sigma_m < \sigma < \sigma_M \tag{59}$$

$$v_{qp}(\sigma) = -v_{pq}(\sigma), \quad \sigma_m < \sigma < \sigma_M. \tag{60}$$

Between two successive transition instants, $p(t)$ and $q(t)$ are both constant: $p(t) \equiv p$, $q(t) \equiv q$; for a force-controlled motion in which $F(t)$ is given, (57) is then a differential equation governing the location $x = s(t)$ of the associated (p, q)-shock.

For definiteness, consider a program of loading in which the given force history $F(t)$, $0 \leqslant t \leqslant T$, begins at $F(0) = 0$, so that initially $\delta(0) = 0$ as well, and suppose that $F(t)$ increases with t to a final value $F(T) > \sigma_M A_M$. To describe the possible quasi-static motions associated with $F(t)$, we shall trace the corresponding force–elongation histories in Figs 6(a) and (b), which contain the information in Figs 4(a) and (b) pertaining to equilibrium

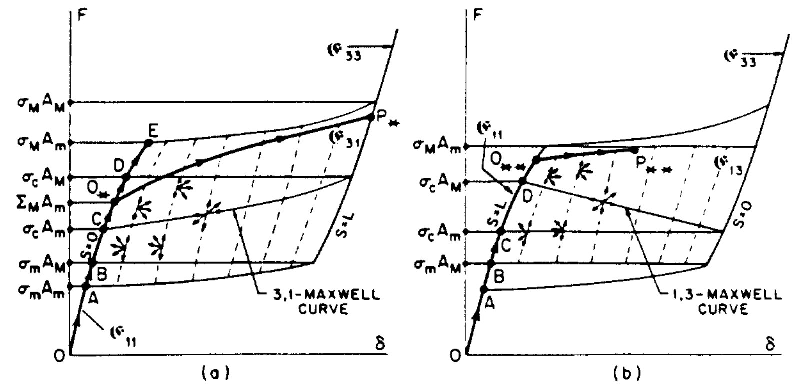


Fig. 6. Possible loading paths in: (a) $\mathfrak{E}_{31}$; (b) $\mathfrak{E}_{13}$.

states with (3, 1)- and (1, 3)-shocks. (Recall from the preceding section that states with either $p = 2$ or $q = 2$, or both, can *never* be reached admissibly during a force history of this kind, so that Figs 4(c)–(f) need not be consulted.) Reference to Fig. 6 shows that, as $F(t)$ increases from zero to the value $\sigma_m A_m$ at, say, time t_1, the associated quasi-static motion is necessarily smooth and of the form (25). During this period, the force elongation relation corresponds to the curve OA in either Fig. 6(a) or (b), and according to (21)–(23), it is given by

$$\delta(t) = \Delta_1(F(t)). \tag{61}$$

As $F(t)$ increases beyond $\sigma_m A_m$, equilibrium states involving (3, 1)-shocks become available (Fig. 6(a)), but as indicated by the arrows in the figure, none can be reached admissibly until the time, say t_2, at which $F(t_2) = \sigma_c A_m$. Thus on the time interval $[t_1, t_2]$, the motion is of the form (27), with $u(x, t) = U_{31}(x;\ F(t), 0) = U_{13}(x;\ F(t), L) = U_1(x;\ F(t))$, the displacement field remains smooth at each instant, (61) remains in force, and the force–elongation curve continues along the arc ABC. For $t > t_2$ and $F(t) > \sigma_c A_m$, the situation changes. Let $F(t_3) = \sigma_c A_M$; when $t_2 < t < t_3$, there are two alternative possibilities, each consistent with admissibility: either the deformation continues to be smooth at each instant, with $u(x, t) = U_1(x; F(t))$, or a (3, 1)-shock is initiated at the end $x = 0$ of the bar at a certain instant $t_* \geqslant t_2$ and begins to advance into the interior in accordance with kinetic relation (54) (or (57)). In the former event, (61) continues to hold, and the force–elongation curve is ABCD (Figs 6(a) and (b)). However, if a (3, 1)-shock forms, then the F–δ curve departs from ABCD at a point O_* (Fig. 6(a)), with

$$\delta = \Delta_{31}(F(t), s(t)). \tag{62}$$

Here $s(t)$ is the location of the shock at time $t > t_*$; it is given by the solution of the differential equation (57) with $p = 3$, $q = 1$, subject to the initial condition $s(t_*) = 0$. If a shock does *not* form and the first of the two above alternatives occurs, the force, having generated only smooth deformations, eventually attains the value $F(t_3) = \sigma_c A_M$, after which any one of *three* mutually exclusive types of admissible response histories may occur. First, the fields may remain smooth, with $\delta(t) = \Delta_1(F(t))$; second, a (3, 1)-shock may emerge at $x = 0$, after which (62) will take over; third, it is now possible to create a (1, 3)-shock at $x = L$. If this third possibility occurs, say at time t_{**}, the force–elongation curve will depart from ABCDE at O_{**} (Fig. 6(b)), and the F–δ relation will be

$$\delta(t) = \Delta_{13}(F(t), s(t)) \tag{63}$$

instead of (61).

The kinetic relation alone does not determine whether or when a shock forms during the above loading program or, if so, whether it is a (3, 1)-shock at $x = 0$ or a (1, 3)-shock at $x = L$. It is therefore necessary to establish in addition a criterion for *shock initiation*. If a (3, 1)-shock forms at $x = 0$ at time t_* (corresponding to the point O_* in Fig. 6(a)), then the strain at the particle $x = 0$ will jump from a value associated with branch 1 of the stress–strain curve to a value on branch 3; for brevity, we say that the particle has undergone a transformation from "phase 1" to "phase 3" at time t_*. The same can be said of the particle $x = L$ if a (1, 3)-shock emerges from $x = L$ at time t_{**} (point O_{**} in Fig. 6(b)). We now adopt a specific criterion for such shock-initiating—or "spontaneous"—phase 1 → phase 3 transformations: the particle $x = x_*$ will spontaneously transform from phase 1 to phase 3 at time t_* if the stress

$$\sigma(x_*, t_*) \geqslant \Sigma_M \text{ and } \sigma(\cdot, t_*) \text{ has a local maximum at } x_*. \tag{64}$$

Here the "transformation stress" Σ_M is determined by the material and presumed to be known. (The alternative to a spontaneous transformation occurs when a particle changes phase due to the passage of a pre-existing shock through that particle.) For the reverse transformation, we similarly postulate that the particle x_* will spontaneously transform from phase 3 to phase 1 at time t_* if

$$\sigma(x_*, t_*) \leqslant \Sigma_m \text{ and } \sigma(\cdot, t_*) \text{ has a local minimum at } x_* \tag{65}$$

where the reverse transformation stress Σ_m is also given. Admissibility requires that Σ_M and Σ_m satisfy

$$\sigma_m < \Sigma_m \leqslant \sigma_c \leqslant \Sigma_M < \sigma_M. \tag{66}$$

Since the bar is monotonically tapered, the maximum stress at each instant occurs at the small end $x = 0$, the minimum at $x = L$. Thus shocks corresponding to phase 1 → phase 3 transformations can only be initiated at $x = 0$, and those corresponding to phase 3 → phase 1 transformations only at $x = L$.

We remark that the shock initiation criterion introduced above can be alternatively formulated in terms of critical values of shock driving *traction*, rather than in terms of the critical values Σ_M and Σ_m of *stress* at the shock. In the present one-dimensional context, no advantage is gained by using this formulation, so we omit it. In higher dimensional settings, however, it is likely that the alternative formulation is essential.

When applied to the loading program described in detail above, our criterion singles out a definite response history from among the possibilities listed there: as t increases from zero, the equilibrium fields remain smooth and the force–elongation relation remains given by (61) until the instant t_* at which $\sigma(0, t_*) = F(t_*)/A_m = \Sigma_M$. At time t_*, a (3, 1)-shock forms at $x = 0$. The evolution of the shock location $s(t)$ is then controlled by the differential equation $(57)_2$ with $p(t) = 3$, $q(t) = 1$, subject to the initial condition $s(t_*) = 0$. Under suitable restrictions on the kinetic response function v_{31} to be specified in the following section, the associated force–elongation response curve, described now by (62), will remain in the set $\mathfrak{s}_{31}$ of (3, 1)-equilibrium states; it is the curve O_*P_* shown schematically in Fig. 6(a).

As the force $F(t)$ increases above the level associated with the point P_*, the subsequent response necessarily is smooth and corresponds to branch 3 strains; thus the F–δ relation now becomes

$$\delta(t) = U_3(F(t), s(t)) \tag{67}$$

during the remainder of the loading process.

If the force $F(t)$ is now *decreased* monotonically to zero from its greatest value $F(T)$, the ensuing deformation will be smooth, and the force–elongation relation will be given by (67) until the minimum stress in the bar $\sigma(L, t)$ reaches the lower transformation stress Σ_m.

At this instant, the particle $x = L$ undergoes a phase 3 → phase 1 transformation, and a (3, 1)-shock is created at $x = L$, moving leftward into the bar in accordance with (57). The F–δ response curve, though again described by (62), will differ from its counterpart during *loading*. Once this curve rejoins OABC, the response during the remainder of the unloading process is that associated with branch 1 smooth fields, continuing down OABC to the origin.

Finally, it should be noted that kinetic relations, the structure of which is substantially more general than (54), could be considered.

7. A SPECIAL CLASS OF KINETIC RELATIONS

We now consider a special class of kinetic response functions V in (54) (or v in (57)). After investigating some of their properties, we illustrate in detail the visco-plastic nature of the macroscopic response of the bar to which they give rise. We discuss rate-independent dissipative response and purely conservative, dissipation-free behavior in the present context as well.

Since we shall be concerned only with equilibrium states accessible through admissible motions from the unloaded, unextended state $F = \delta = 0$, we will not encounter strains on branch 2 of the stress–strain curve. Moreover, for the loading programs to be considered, the shock initiation criterion of the preceding section will always rule out (1, 3)-shocks. Thus we shall be concerned with kinetic relation (54) only when $p(t) = 3$, $q(t) = 1$. As a result, we shall write $V_{31} \equiv V$, $v_{pq} \equiv v$, $M_{31} \equiv M$, $m_{31} \equiv m$ throughout the present section for convenience. Recall from (46), (47) and (53) that $m < 0$, $M > 0$.

7.1. *The kinetic response function*

Guided in part by the form sometimes ascribed to the counterpart of function V in microstructural theories of plasticity (Martin, 1975), we now make three assumptions concerning the form of V. (i) We assume that there are numbers m_* and M_* such that

$$m < m_* \leqslant 0 \leqslant M_* < M \tag{68}$$

and

$$\left.\begin{aligned} V(f) &< 0 \quad \text{for} \quad m < f < m_* \\ V(f) &= 0 \quad \text{for} \quad m_* \leqslant f \leqslant M_* \\ V(f) &> 0 \quad \text{for} \quad M_* < f < M \end{aligned}\right\} \tag{69}$$

noting that (69) is consistent with requirement (55) imposed by admissibility. (ii) We assume that $V(f)$ is continuous on (m, M) and continuously differentiable on $(m, m_*] + [M_*, M)$, and that $V'(f) > 0$ for $m < f < m_*$ and for $M_* < f < M$. (iii) Finally, we assume that $V(f)$ is unbounded as $f \to m+$ as $f \to M-$; more precisely, we require that

$$\left.\begin{aligned} V(f) &\leqslant C_m(f-m)^{-1} \text{ for } f \text{ sufficiently near } m \\ V(f) &\geqslant C_M(M-f)^{-1} \text{ for } f \text{ sufficiently near } M \end{aligned}\right\} \tag{70}$$

for suitable constants $C_m < 0$ and $C_M > 0$.

According to (i), a (3, 1)-shock will move in the $+x$-direction only if the shock driving traction exceeds M_*, and in the reverse direction only if f is less than m_*. Permitting $V(f)$ to vanish over the interval $[m_*, M_*]$ will be seen later to allow for the possibility of dissipation-free unloading and re-loading. Assumption (ii) assures that, roughly speaking, larger shock tractions promote greater shock speeds. The role of property (70) will become clear as we proceed. A schematic sketch of the graph of $V(f)$ is shown in Fig. 7(a); it is reminiscent of the relationship sometimes proposed between the driving force on a dislocation and dislocation velocity (Martin, 1975).

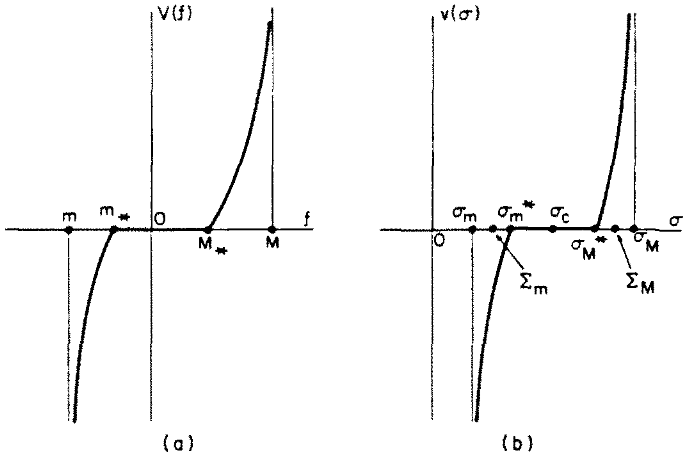


Fig. 7. Kinetic response functions V and v.

By (58), property (69) of V translates into a corresponding property of v

$$\left.\begin{aligned} v(\sigma) &< 0 \quad \text{for} \quad \sigma_m < \sigma < \sigma_m^* \\ v(\sigma) &= 0 \quad \text{for} \quad \sigma_m^* \leqslant \sigma \leqslant \sigma_M^* \\ v(\sigma) &> 0 \quad \text{for} \quad \sigma_M^* < \sigma < \sigma_M \end{aligned}\right\} \tag{71}$$

where σ_m^* and σ_M^* are the unique numbers defined by

$$f_{31}(\sigma_m^*) = m_*, \quad f_{31}(\sigma_M^*) = M_*. \tag{72}$$

Clearly, σ_m^* and σ_M^* satisfy

$$\sigma_m < \sigma_m^* \leqslant \sigma_c \leqslant \sigma_M^* < \sigma_M \tag{73}$$

where σ_c is the Maxwell stress (Fig. 1). Clearly, the two transformation stresses Σ_M and Σ_m associated with shock initiation must satisfy

$$\sigma_m < \Sigma_m \leqslant \sigma_m^* \leqslant \sigma_c \leqslant \sigma_M^* \leqslant \Sigma_M < \sigma_M. \tag{74}$$

Property (70) of V is readily shown to imply that there are numbers σ_m' and σ_M', with $\sigma_m < \sigma_m' < \sigma_m^*$ and $\sigma_M^* < \sigma_M' < \sigma_M$, and such that

$$\left.\begin{aligned} v(\sigma) &\leqslant c_m(\sigma-\sigma_m)^{-1} \quad \text{for} \quad \sigma_m < \sigma < \sigma_m' \\ v(\sigma) &\geqslant c_M(\sigma_M-\sigma)^{-1} \quad \text{for} \quad \sigma_M' < \sigma < \sigma_M \end{aligned}\right\} \tag{75}$$

for suitable constants $c_m < 0$ and $c_M > 0$. A schematic graph of $v(\sigma)$ is shown in Fig. 7(b).

If φ stands for the inverse of the restriction of v to $(\sigma_m, \sigma_m^*] + [\sigma_M^*, \sigma_M)$, version (57) of the kinetic relation may be put in the alternative form

$$\bar{\sigma}(t) = F(t)/A(s(t)) = \varphi(\dot{s}(t)). \tag{76}$$

Note that φ has a discontinuity at the origin unless $\sigma_m^* = \sigma_c = \sigma_M^*$.

Suppose for the moment that we have a motion taking place on the time interval $[t_0, T]$ and involving a (3, 1)-shock located at $x = s(t)$ at time t. As time increases, the shock location $s(t)$ will evolve according to the kinetic relation (57), which in present notation is

$$\dot{s}(t) = v(\bar{\sigma}(t)) = v(F(t)/A(s(t))). \tag{77}$$

Property (75) guarantees that the moving point $(s(t), F(t))$ in the (s, F)-plane remains in the set $\mathscr{S}_{31}$ corresponding to equilibrium states with (3, 1)-shocks. To prove this, it is sufficient to show that the stress at the shock $\bar{\sigma}(t) = F(t)/A(s(t))$ never exceeds σ_M and is never less than σ_m. Suppose that, at the initial instant t_0, one has $\sigma'_M < \bar{\sigma}(t_0) < \sigma_M$, so that the inequality $(75)_2$ holds at time t_0. We shall show that $\bar{\sigma}(t) < \sigma_M$ for *all* t in $[t_0, T]$. Suppose this were not the case. Then there would be instants t in $(t_0, T]$ at which $\bar{\sigma}(t) = \sigma_M$; let t_{1*} be the infimum of all such instants. Clearly

$$t_0 < t_{1*} \leqslant T, \quad \bar{\sigma}(t) < \sigma_M \quad \text{for} \quad t_0 \leqslant t < t_{1*}, \quad \text{and} \quad \bar{\sigma}(t_{1*}) = \sigma_M \tag{78}$$

by the continuity of $\bar{\sigma}(t)$. Now let t_{0*} be the supremum of the set of all times t for which $\bar{\sigma}(t) \leqslant \sigma'_M$ and $t_0 < t < t_{1*}$. Then for $t_{0*} < t < t_{1*}$, we have $\sigma'_M < \bar{\sigma}(t) < \sigma_M$, so that inequality $(75)_2$ applies, and $v(\bar{\sigma}(t)) > 0$, whence by (77), $\dot{s}(t) > 0$ during this time interval as well. It follows that we may express t as a function of $s: t = \hat{t}(s)$, and thus regard $\bar{\sigma}(\hat{t}(s)) = \bar{\sigma}(s)$ as a function of s as well. Then by (41) and (77)

$$\begin{aligned} \bar{\sigma}'(s)A(s) + \bar{\sigma}(s)A'(s) = \mathrm{d}/\mathrm{d}s\{F(\hat{t}(s))\} &= \dot{F}(\hat{t}(s))/\dot{s}(\hat{t}(s)) \\ &= \dot{F}(\hat{t}(s))/v(\bar{\sigma}(s)). \end{aligned} \tag{79}$$

Let $\lambda = \max |\dot{F}(t)|$, $t_0 \leqslant t \leqslant T$, be the maximum loading rate during the motion. Then by $(75)_2$ and (79)

$$\bar{\sigma}'(s)A(s) + \bar{\sigma}(s)A'(s) < (\lambda/c_M)(\sigma_M - \bar{\sigma}(s)), \quad s_{0*} < s < s_{1*} \tag{80}$$

where $s_{0*} = s(t_{0*})$, $s_{1*} = s(t_{1*})$. Integration of this linear differential inequality and using the fact that $A(s_{0*}) \leqslant A(s)$ leads to

$$\bar{\sigma}(s) \leqslant \sigma_M - (\sigma_M - \sigma_0) \exp\left\{-(\lambda/c_M)\int_{s_{0*}}^{s} A(x)^{-1}\,\mathrm{d}x\right\}, \quad s_{0*} \leqslant s \leqslant s_{1*} \tag{81}$$

with $\sigma_0 = \bar{\sigma}(s_{0*})$. In particular, this gives $\bar{\sigma}(s_{1*}) = \bar{\sigma}(t_{1*}) < \sigma_M$, contradicting (78). It follows that $\bar{\sigma}(t) < \sigma_M$ for $t_0 \leqslant t \leqslant T$. Thus the stress $\bar{\sigma}(t)$ at the shock in a motion governed by (77) never exceeds σ_M; a similar argument shows that $\bar{\sigma}(t)$ is never less than σ_m.

7.2. *Macroscopic response*

We now elucidate the nature of the macroscopic response of the bar under various force-controlled programs of loading for the class of kinetic relations described above. First, let $F(t) = \lambda t$, $0 \leqslant t \leqslant T$, corresponding to loading at a constant rate λ from the undeformed state. Assume that the final value of force is such that $F(T) > \sigma_M A_M$. The force–elongation response answering to this loading is shown schematically in Fig. 8(a). After loading begins, the point $(\delta(t), F(t))$ rises from the origin O along the response curve OAO_* associated with smooth fields of the type (25), so that $\delta(t) = \Delta_1(F(t))$, and there is no dissipation. When the force reaches the level corresponding to the point O_* in Fig. 8(a), the stress at $x = 0$ coincides with the larger transformation stress Σ_M, and a (3, 1)-shock is created at $x = 0$ according to the shock initiation criterion of the preceding section. Kinetic relation (77) takes over, and the initial shock velocity has the value $v(\Sigma_M)$. By (74), $\Sigma_M \geqslant \sigma^*_M$, so that by (71), $v(\Sigma_M) \geqslant 0$. If $\Sigma_M > \sigma^*_M$, then $v(\Sigma_M) > 0$, the initial shock velocity is positive, and there is a discontinuity in slope in the F–δ response curve at O_*, as shown in the figure. If $\Sigma_M = \sigma^*_M$, the initial shock velocity is zero, and the slope of the F–δ curve is continuous at O_*. Under the control of (77), $s(t)$ increases with t, the elongation is given by $\delta(t) = \Delta_{31}(F(t), s(t))$, and the point $(\delta(t), F(t))$ moves along the curve $\mathrm{O}_*\mathrm{BP}$, both $F(t)$ and $\delta(t)$ increasing. This stage of the process is accompanied by dissipation. When the force has increased to the value associated with point P in Fig. 8(a), the shock has arrived at the end $x = L$ of the bar, all particles are in phase 3, and the field is smooth. As the force continues to increase, the response is that of smooth, dissipation-free phase 3 deformations, with $\delta(t) = \Delta_{33}(F(t))$. The loading terminates at time T, corresponding to point Z in the figure.

Now suppose that $F(t)$ is decreased at the constant rate λ from its largest value $F(T)$ to zero. The response curve at first follows the arc ZPO_* corresponding to smooth phase 3 fields, and the response is dissipation free; at Q_*, the stress at the larger end $x = L$ of the bar has diminished to the smaller transformation stress Σ_m, and a leftward moving (3, 1)-shock emerges. The sign of v in (77) is now negative, $s(t)$, $F(t)$ and $\delta(t)$ all decrease, and the arc Q_*CA is traced out as the shock returns to $x = 0$, dissipating as it moves. As $F(t)$ decreases to zero from its value at A, $\delta(t) = \Delta_1(F(t))$ again along the arc AO, and the final stage of unloading takes place without dissipation. The *total* dissipation in the loading cycle is of course precisely the area of the hysteresis loop AO_*PQ_*A.

If the loading rate λ were changed, the loading and unloading "yield points" O_* and Q_* would remain the same, but the arcs O_*BP and Q_*CA associated with the dissipative portion of the process would change. The macroscopic response is thus rate dependent.

Consider now a modified version of the loading history described above in which the force $F(t)$, after arriving at the value associated with point B in Fig. 8(a), is decreased, and then ultimately increased again. Figure 8(b) shows the resulting macroscopic response. As before, the response curve is the arc OO_*B during the initial loading phase, the portion O_*B being dissipative. When $F(t)$ begins to decrease, $v(F(t)/A(s(t)))$ remains positive at first, and (77) requires $s(t)$ to continue to increase, accompanied by dissipation. During this stage, $\delta(t)$ will also continue to increase, generating the arc BC of the response curve. At point C, the stress $\bar{\sigma}$ at the shock has dropped to the value σ_M^*, so that by (71), $v(\bar{\sigma}) = 0$ at the corresponding instant. If $\sigma_M^* > \sigma_m^*$ in (71), and if $F(t)$ continues to decrease below its value at C, $\bar{\sigma}(t)$ will remain in the range for which $v(\bar{\sigma}(t))$ vanishes, so that $\dot{s}(t) = 0$ and the shock remains stationary. During this portion of the unloading process, the corresponding part CD of the response curve lies along a curve of constant s, and there is no dissipation. If now the force $F(t)$ is increased, the initial portion of the reloading process takes place along DC and is dissipation free. If the force ultimately increases sufficiently to raise the stress at the shock to a value greater than σ_M^*, the shock resumes its motion, dissipation will begin again, and the response curve will proceed along CE. This particular force history illustrates the occurrence of dissipation-free unloading with a stationary shock.

If, during unloading in the last program, the force had been decreased sufficiently below its value at D, the stress at the shock would diminish below the value σ_m^*, causing

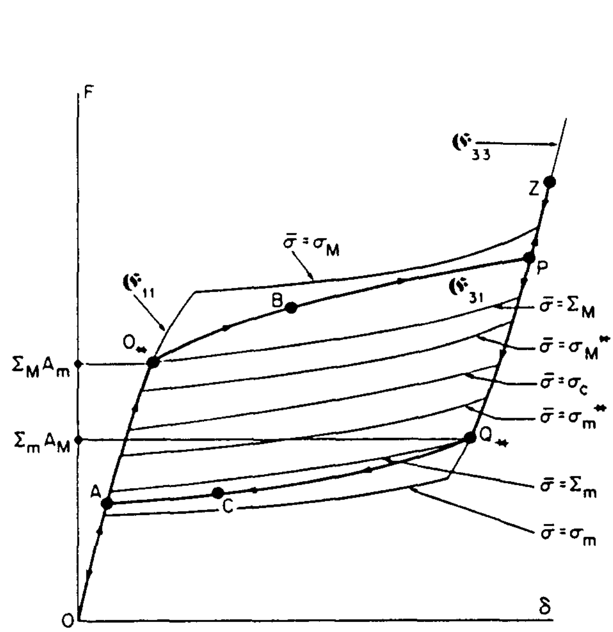


Fig. 8(a). Loading–unloading path according to kinetic relation (77).

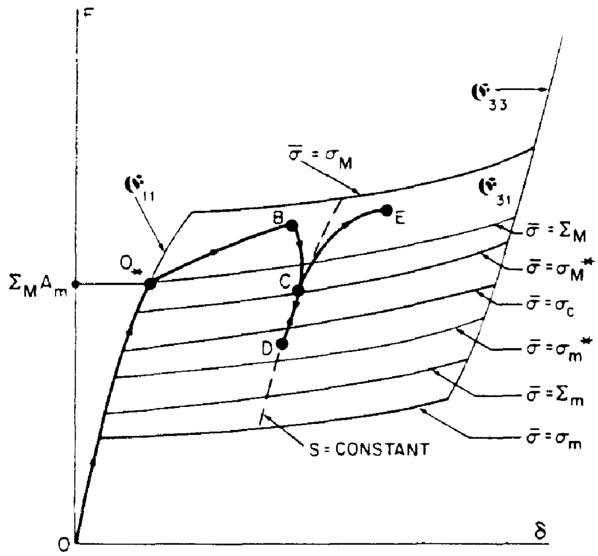


Fig. 8(b). First loading unloading reloading path according to kinetic relation (77).

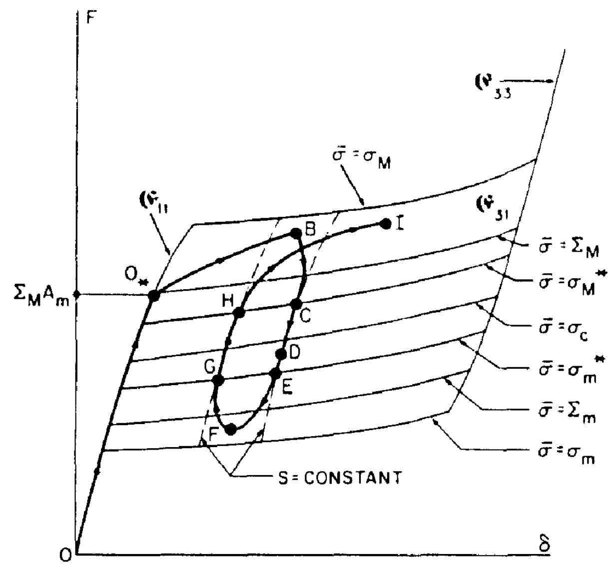


Fig. 8(c). Second loading–unloading–reloading path according to kinetic relation (77).

$v(\bar{\sigma}(t))$ to become negative, forcing the shock to move to the left. Figure 8(c) shows the macroscopic response curve OO_*BFI for such a force history, together with the response on reloading. The arcs OO_*, CE, and GH correspond to dissipation-free periods during the quasi-static motion.

The macroscopic response of the bar during the loading programs just described clearly resembles that associated with visco-plastic behavior in several respects. One feature of the latter kind of behavior that is not present here is that of permanent strain. By abandoning the requirement $\sigma_m > 0$ in (9) and thus considering a stress–strain curve the local minimum of which (Fig. 1) corresponds to a compressive stress σ_m, one can introduce permanent strain into the macroscopic response (Abeyaratne and Knowles, 1987b).

7.3. *Rate-independent behavior*

The form of the kinetic response function sketched in Fig. 7(b) suggests consideration of the limiting case in which the function φ inverse to v is a step function as shown in Fig. 9:

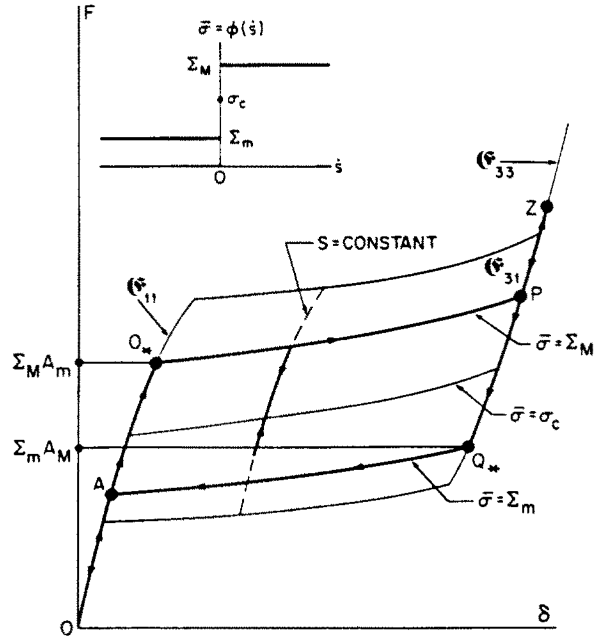


Fig. 9. Rate-independent macroscopic response.

$$\varphi(\dot{s}) = \begin{cases} \Sigma_M & \text{for} \quad \dot{s} > 0 \\ \Sigma_m & \text{for} \quad \dot{s} < 0 \end{cases} \tag{82}$$

where Σ_M and Σ_m are the shock initiation stresses, $\sigma_m \leqslant \Sigma_m < \sigma_c < \Sigma_M \leqslant \sigma_M$ and σ_c is the Maxwell stress. One shows readily that the macroscopic response produced by the kinetic relation is rate independent and is of the form shown in Fig. 9. If $\Sigma_m = \sigma_m$ and $\Sigma_M = \sigma_M$, the quasi-static motions permitted by the kinetic relation are *maximally dissipative* in a definite sense (response of this kind is discussed in detail in Abeyaratne and Knowles (1987a)).

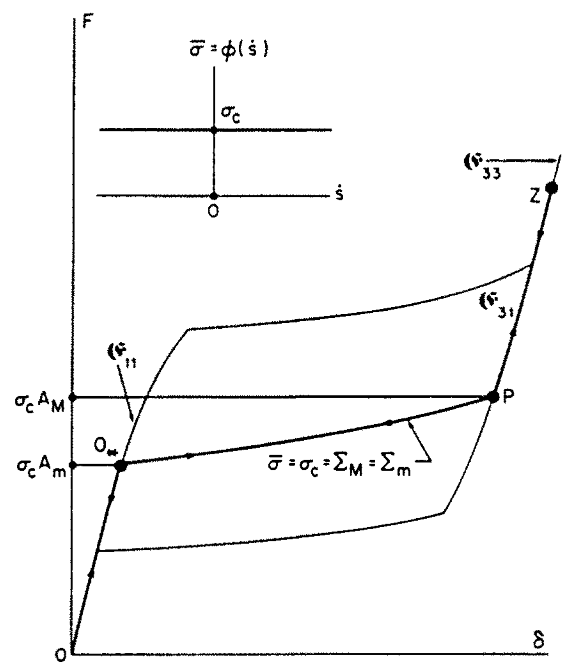


Fig. 10. Dissipation-free macroscopic response.

7.4. *Dissipation-free macroscopic response*

Finally, we note that the purely conservative (or dissipation free) response of the kind conventionally associated with elastic behavior results from choosing the inverse kinetic response function φ to be

$$\varphi(\dot{s}) = \sigma_c \quad \text{for} \quad -\infty < \dot{s} < \infty \tag{83}$$

and taking both shock initiation stresses Σ_M and Σ_m to be equal to the Maxwell stress σ_c. In this case, the macroscopic response is independent of past history and is as shown in Fig. 10.

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REFERENCES

Abeyaratne, R. (1980). An admissibility condition for equilibrium shocks in finite elasticity. *J. Elasticity* **13**, 175–184.

Abeyaratne, R. (1983). Discontinuous deformation gradients in the finite twisting of an incompressible elastic tube. *J. Elasticity* **11**, 42–80.

Abeyaratne, R. and Knowles, J. K. (1987a). Non-elliptic elastic materials and the modeling of dissipative mechanical behavior: an example. *J. Elasticity* **18**, 227–278.

Abeyaratne, R. and Knowles, J. K. (1987b). Non-elliptic elastic materials and the modeling of elastic–perfectly plastic behavior for finite deformation. *J. Mech. Phys. Solids* **35**, 343–365.

Ball, J. M. and James, R. D. (1987). Fine phase mixtures as minimizers of energy. *Arch. Ration. Mech. Analysis* **100**, 13–52.

Budiansky, B., Hutchinson, J. W. and Lambropoulos, J. (1983). Continuum theory of dilatant transformation toughening in ceramics. *Int. J. Solids Structures* **19**, 337–355.

Delaey, L., Krishnan, R. V., Tas, H. and Warlimont, H. (1974). Review: Thermoelasticity, pseudo-elasticity and the memory effects associated with martensitic transformations. *J. Mater. Sci.* **9**, 1521–1555.

Ericksen, J. L. (1975). Equilibrium of bars. *J. Elasticity* **5**, 191–201.

Eshelby, J. D. (1956). The continuum theory of lattice defects. In *Solid State Physics* (Edited by F. Seitz and D. Turnbull), Vol. 3. Academic Press, New York.

Eshelby, J. D. (1970). Energy relations and the energy-momentum tensor in continuum mechanics. In *Inelastic Behavior of Solids* (Edited by M. F. Kanninen *et al.*). McGraw-Hill, New York.

Fosdick, R. L. and James, R. D. (1981). The elastica and the problem of pure bending for a non-convex stored energy function. *J. Elasticity* **11**, 165–186.

Fosdick, R. L. and MacSithigh, G. (1983). Helical shear of an elastic circular tube with a non-convex stored energy. *Arch. Ration. Mech. Analysis* **84**, 31–53.

Gurtin, M. E. (1983). Two-phase deformations of elastic solids. *Arch. Ration. Mech. Analysis* **84**, 1–29.

James, R. D. (1979). Co-existent phases in the one-dimensional static theory of elastic bars. *Arch. Ration. Mech. Analysis* **72**, 99–140.

James, R. D. (1981). Finite deformation by mechanical twinning. *Arch. Ration. Mech. Analysis* **77**, 143–176.

James, R. D. (1986). Displacive phase transformations in solids. *J. Mech. Phys. Solids* **34**, 359–394.

Knowles, J. K. (1979). On the dissipation associated with equilibrium shocks in finite elasticity. *J. Elasticity* **9**, 131–158.

Knowles, J. K. and Sternberg, E. (1978). On the failure of ellipticity and the emergence of discontinuous deformation gradients in plane finite elastostatics. *J. Elasticity* **8**, 329–379.

Landau, L. D. and Lifshitz, E. M. (1959). *Fluid Mechanics*, Vol. 6 of *Course of Theoretical Physics*. Pergamon Press, New York.

Martin, J. B. (1975). *Plasticity: Fundamentals and General Results*. MIT Press, Cambridge, Massachusetts.

Rice, J. R. (1970). On the structure of stress–strain relations for time-dependent plastic deformation in metals. *J. Appl. Mech.* **37**, 728–737.

Rice, J. R. (1971). Inelastic constitutive relations for solids: an internal-variable theory and its application to metal plasticity. *J. Mech. Phys. Solids* **19**, 433–455.

Rice, J. R. (1975). Continuum mechanics and thermodynamics of plasticity in relation to microscale deformation mechanisms. In *Constitutive Equations in Plasticity* (Edited by A. S. Argon), pp. 23–79. MIT Press, Cambridge, Massachusetts.

Silling, S. A. (1988). Consequences of the Maxwell relation for anti-plane shear deformations of an elastic solid. *J. Elasticity* **19**, 241–284.